

INVARIANT MEASURES FOR THE TWO-DIMENSIONAL AVERAGED-EULER EQUATIONS

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ABSTRACT. We define a Gaussian invariant measure for the two-dimensional averaged-Euler equation and show the existence of its solution with initial conditions on the support of the measure. An invariant surface measure on the level sets of the energy is also constructed, as well as the corresponding flow. Uniqueness is proved. Poincaré recurrence theorem is used to show that the flow returns infinitely many times in a neighborhood of the initial state.

CONTENTS

1. Introduction	1
2. The averaged-Euler equations	3
2.1. Conserved quantities of the motion	4
2.2. Fourier expansion of the system	5
3. Gaussian invariant measures for the averaged-Euler equations	6
3.1. Regularity of the vector field	7
3.2. The vector field is divergence free	11
4. A surface measure	15
5. The invariant flow	18
5.1. Existence	18
5.2. Uniqueness	22
5.3. Return to a neighborhood of its initial state	24
References	24

1. INTRODUCTION

The purpose of this paper is to built invariant measures for the averaged-Euler equations. The averaged-Euler equations were introduced in 1998 by D. D. Holm, J. E. Marsden and T. S. Ratiu in [10]. For an incompressible non-viscous fluid the equations are the following

$$\frac{\partial Au}{\partial t} + (u \cdot \nabla)Au + (\nabla u)^T \cdot Au = -\nabla p, \quad \nabla \cdot u = 0,$$

where $A = (1 - a^2 \Delta)^s$ for a a real parameter and s a positive number. The mean velocity of the flow is denoted by $u : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and the pressure by $p : \mathbb{R}^2 \rightarrow \mathbb{R}$. The authors of [10] consider a modification of the Euler equations such that non linear effects at small

scales of the motion are negligible; therefore the dynamics remains turbulent, but non dissipative.

Global existence and uniqueness for solutions of the two-dimensional averaged-Euler equation are known both in \mathbb{R}^2 and in a bounded domain for initial velocities in H^3 , see respectively V. Busuioc [6] and S. Shkoller [15]. In the latter reference the classical pde problem is transformed into a geometric one, considering geodesics in the (infinite-dimensional) group of volume-preserving diffeomorphisms. Averaged-Euler equations are indeed known to describe the velocities of geodesics on this group endowed with the H^1 metric. For further results concerning these equations we cite [7] and references therein.

For this system O. Bell, A. Chorin and W. Crutchfield pointed out in [5] that invariant Gibbs measures can be considered, as the equations conserve the energy and the enstrophy. In their perspective the invariant measures are used to perform numerical predictions of the dynamics. Invariant measures are of significant interest when employed to improve existing deterministic results, particularly when we deal with vector fields with low regularity and may be used, among others, to extend local to global existence results or to prove recurrence properties for a flow, see for example [17]. Also quasiinvariant measures may serve to the same purposes and besides they are sometimes supported on more regular spaces, see for example [16].

We consider an equivalent formulation, the vorticity formulation, of the averaged-Euler equations. On \mathbb{R}^2 , for divergence free velocity fields, the “stream function” $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$ is defined by $u = \nabla^\perp \varphi := (-\partial_2 \varphi, \partial_1 \varphi)$. The vorticity formulation in terms of stream function is the following

$$\frac{\partial A \Delta \varphi}{\partial t} + (\nabla^\perp \varphi \cdot \nabla) A \Delta \varphi = 0. \quad (1)$$

Below we consider (1) on $\mathbb{T}^2 \simeq [0, 2\pi] \times [0, 2\pi]$ and with periodic boundary conditions.

In this paper we rigorously define for this system an infinite dimensional Gibbs measure with respect to the enstrophy, formally

$$d\mu_\gamma \simeq \frac{1}{Z} e^{-\gamma \frac{Enstrophy}{2}} d\lambda,$$

where $d\lambda$ denotes “Lebesgue measure”, Z a normalizing constant and γ is a positive parameter. We construct a flow for the averaged-Euler equations on the support of this measure. Namely, as previously done by A. B. Cruzeiro and S. Albeverio in [2] for the analogous case of the Euler equations, applying a combination of Prohorov and Skorohod’s theorems to finite dimensional flow approximations, we can construct continuous flows for the averaged-Euler vector field on some probability space $(\Omega, \mathcal{F}, P_\gamma)$ with values in $H^{1-\alpha, s}$ for some $s > 0$ and $\alpha > 2$, that is in a Sobolev space of negative order. Therefore these pointwise continuous flows belong to a distribution space. In particular we will have

$$U(t, \omega) = U(0, \omega) + \int_0^t B(U(s, \omega)) ds, \quad P_\gamma - a.e. \ \omega \in \Omega, \ \forall t \in \mathbb{R},$$

with μ_γ invariant under the flow. See Theorem 3.2 below.

We also consider, as previously done for the Euler equations by F. Cipriano in [8], the infinite-dimensional conditional measure defined on level sets of the energy. Comparing

with the Euler case, there is no need here to define a renormalized energy, since in the averaged-Euler case the energy itself is square integrable with respect to μ_γ . This surface measure ν_γ^r , is also invariant and therefore pointwise continuous flows can be constructed on a probability space with values on the level sets of the energy, say V^r , for some positive r . We have

$$U'(t, \omega) = U'(0, \omega) + \int_0^t B^*(U'(s, \omega)) ds, \quad P_\gamma^r - a.e. \omega, \quad \forall t \in \mathbb{R},$$

where B^* is any redefinition of B . Moreover ν_γ^r is invariant under the flow. See Theorem 5.2 below.

Uniqueness holds in both cases considered above. Indeed we are in a particular case of [4] in which L. Ambrosio and A. Figalli extended to the infinite-dimensional case the approach *à la* DiPerna-Lions for the theory of flows associated to weakly differentiable vector fields, where uniqueness holds. In particular we can say that for every fixed initial data the laws of the constructed flows are Dirac masses. This implies, on one hand, that the respective flows are undistinguishable and in this sense unique, on the other that our solution is in fact deterministic. Meaning, respectively for Theorem 3.2 and 5.2, that the following strongest expressions hold

$$U(t, \varphi) = U(0, \varphi) + \int_0^t B(U(s, \varphi)) ds, \quad \mu_\gamma - a.e. \varphi \in H^{1-\alpha, s}, \quad \forall t \in \mathbb{R},$$

and

$$U'(t, \varphi) = U'(0, \varphi) + \int_0^t B^*(U'(s, \varphi)) ds, \quad \nu_\gamma^r - a.e. \varphi \in V^r \subset H^{1-\alpha, s}, \quad \forall t \in \mathbb{R},$$

see Remark 5.1 below.

Finally since the Poincaré recurrence theorem holds, we have that the flow returns to a neighborhood of the initial state infinitely many times. The analogous for the Euler system was proved in [9] by A. Constantin and D. Levy.

In Section 2 we define the spaces of functions which are relevant in our work; we rewrite the vorticity formulation for the averaged-Euler equations as an infinite dimensional system of ordinary differential equations using the Fourier coefficients of the stream function. Here we also show that the energy and the enstrophy are conserved quantities. In Section 3 we rigorously define a Gibbs measure μ_γ and describe its support. Moreover we study the $L_{\mu_\gamma}^p$ regularity of the vector field and we show that it is divergence free with respect to μ_γ . Finally we construct a flow on a suitable probability space. In Section 4 we define the infinite dimensional conditional measure ν_γ^r with support on the level sets of the energy V^r . In Section 5 we show that ν_γ^r is invariant and prove existence and uniqueness of a flow defined on some probability space. As a byproduct of the proof of the uniqueness we conclude that the solution holds in a stronger sense. Finally we show that the solution returns to a vicinity of the initial state infinitely many times.

2. THE AVERAGED-EULER EQUATIONS

Consider the operator $A = (1 - a^2 \Delta)^s$ for a a real parameter and s a positive number; if s is not an integer, A is a pseudo-differential operator. The averaged-Euler equations

for an incompressible non-viscous fluid on \mathbb{R}^2 are (c.f. [5], [10])

$$\frac{\partial Au}{\partial t} + (u \cdot \nabla)Au + (\nabla u)^T \cdot Au = -\nabla p, \quad \nabla \cdot u = 0 \quad (2)$$

where $u : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is the velocity of the flow and $p : \mathbb{R}^2 \rightarrow \mathbb{R}$ is the pressure. In what follows we denote $\nabla^\perp = (-\partial_2, \partial_1)$ where ∂_1, ∂_2 are the partial derivatives with respect to the first and second variable.

We have the following

Theorem 2.1. *A time dependent vector field u is a smooth solution of (2) if and only if there exists a smooth (real) function φ such that $u = \nabla^\perp \varphi$ and φ is a solution of the equation*

$$\frac{\partial A\Delta\varphi}{\partial t} + (\nabla^\perp \varphi \cdot \nabla)A\Delta\varphi = 0. \quad (3)$$

Proof. Taking the “curl” of (2),

$$\frac{\partial A\nabla^\perp \cdot u}{\partial t} + \nabla^\perp \cdot [(u \cdot \nabla)Au] + \nabla^\perp \cdot [(\nabla u)^T \cdot Au] = 0,$$

we get

$$\frac{\partial A\nabla^\perp \cdot u}{\partial t} + (u \cdot \nabla)A\nabla^\perp \cdot u = 0.$$

From the condition $\nabla \cdot u = 0$ we know that exists a real-valued function φ , called the stream function, such that $u = \nabla^\perp \varphi$; thus sufficiency is proved. To prove necessity let f be defined by

$$f = -\frac{\partial A\nabla^\perp \varphi}{\partial t} - (\nabla^\perp \varphi \cdot \nabla)A\nabla^\perp \varphi - (\nabla \nabla^\perp \varphi)^T \cdot A\nabla^\perp \varphi.$$

Taking the “curl” we get

$$-\frac{\partial A\Delta\varphi}{\partial t} - (\nabla^\perp \varphi \cdot \nabla)A\Delta\varphi = \nabla^\perp \cdot f,$$

then $\nabla^\perp \cdot f = 0$ by assumption and thus there exists a scalar function p such that $f = \nabla p$. The proof is performed in detail in [3] for the analogous case of the Euler system. \square

Our general settings will be similar to the ones in [8, 2, 3] where the case of Euler equation is studied. We will consider our equations on the two-dimensional torus $\mathbb{T}^2 \simeq [0, 2\pi] \times [0, 2\pi]$ and with periodic boundary conditions, that is

$$\varphi(0, y, t) = \varphi(2\pi, y, t) \text{ and } \varphi(x, 0, t) = \varphi(x, 2\pi, t), \quad \forall (x, y) \in \mathbb{T}^2.$$

Remark 2.1. From the expression of the vorticity equation we remark that if $s = 0$ we are considering the Euler system.

2.1. Conserved quantities of the motion. The averaged-Euler equation is conservative, meaning that the “energy” (u, Au) is an invariant of the motion (the inner product is the one of $L^2(\mathbb{T}^2)$); also the “enstrophy” $(A\nabla^\perp \cdot u, A\nabla^\perp \cdot u)$ is a conserved quantity and we can write these quantities in terms of the stream function φ as

$$E = -\frac{1}{2} \int_{\mathbb{T}^2} \varphi A\Delta\varphi dx$$

and

$$S = \frac{1}{2} \int_{\mathbb{T}^2} (A\Delta\varphi)^2 dx.$$

We have in fact that

$$\begin{aligned} \frac{dE}{dt} &= - \left(\frac{\partial}{\partial t} A\Delta\varphi, \varphi \right) = \left((\nabla^\perp \varphi \cdot \nabla) A\Delta\varphi, \varphi \right) \\ &= \left(\nabla^\perp \varphi \cdot \nabla \varphi, A\Delta\varphi \right) = 0 \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dt} S &= \left(\frac{\partial A\Delta\varphi}{\partial t}, A\Delta\varphi \right) = - \left((\nabla^\perp \varphi \cdot \nabla) A\Delta\varphi, A\Delta\varphi \right) \\ &= - \left(\nabla^\perp \varphi \cdot \nabla \varphi, A\Delta A\Delta\varphi \right) = 0 \end{aligned}$$

since $\nabla^\perp \varphi \cdot \nabla \varphi = 0$.

2.2. Fourier expansion of the system. We want to write the averaged-Euler partial differential equation as an infinite dimensional ordinary differential equation by means of Fourier expansion series (c.f. [2, 3] for an analogous formulation of the Euler equation). We consider an orthonormal basis of $L^2(\mathbb{T}^2)$, $\{e_k(x)\}_{k \in \mathbb{Z}^2}$, defined by $e_k(x) = \frac{1}{2\pi} e^{ik \cdot x}$. These functions are eigenfunctions of the Laplace operator:

$$\Delta e_k(x) = -k^2 e_k(x), \quad \forall k \in \mathbb{Z}^2.$$

Here $k \cdot x = k_1 x_1 + k_2 x_2$ for $k = (k_1, k_2) \in \mathbb{Z}^2$ and $x = (x_1, x_2) \in \mathbb{T}^2$ and $k^2 = k \cdot k$. We say that $k \in \mathbb{Z}^2$ is positive if $k_1 > 0$ or $k_1 = 0$ and $k_2 > 0$. The Sobolev spaces

$$\mathcal{H}^{2s+2}(\mathbb{T}^2) = \left\{ v : \mathbb{T}^2 \rightarrow \mathbb{R} : \int \sum_{|\alpha| \leq 2s+2} |D^\alpha v(x)|^2 dx < +\infty \right\}$$

can be identified with the complex Hilbert spaces

$$H^{2,s} = \left\{ v = \sum_{k \in \mathbb{Z}^2} v_k e_k : \sum_{k > 0} k^4 (1 + a^2 k^2)^{2s} |v_k|^2 < +\infty \right\}$$

with inner product $\langle u, v \rangle_{2,s} = \sum_{k > 0} k^4 (1 + a^2 k^2)^{2s} u_k \bar{v}_k$. For general $p \in \mathbb{R}$ we define

$$H^{p,s} = \left\{ v = \sum_{k \in \mathbb{Z}^2} v_k e_k : \sum_{k > 0} k^{2p} (1 + a^2 k^2)^{ps} |v_k|^2 < +\infty \right\}$$

with inner product $\langle u, v \rangle_{p,s} = \sum_{k > 0} k^{2p} (1 + a^2 k^2)^{ps} u_k \bar{v}_k$.

Henceforth we write $\varphi(x, t) = \sum_{h > 0} \omega_h(t) e_h(x)$ and we write the energy and the enstrophy as

$$E = \frac{1}{2} \|\varphi\|_{1,s}^2$$

and

$$S = \frac{1}{2} \|\varphi\|_{2,s}^2.$$

Set $h'^\perp = (-h'_2, h'_1)$ for $h' \in \mathbb{Z}^2$; we have

$$\begin{aligned} \nabla^\perp \varphi \cdot \nabla A \Delta \varphi &= -\frac{1}{2\pi} \sum_{\substack{h>0, \\ h'>0, \\ h' \neq h}} \omega_h \omega_{h'} (h \cdot h'^\perp) h'^2 (1 + a^2 h'^2)^s e_{h+h'}(x) \\ &= -\frac{1}{2\pi} \sum_{\substack{h>0, \\ k>0, \\ h'+h=k}} \omega_h \omega_{h'} (h \cdot h'^\perp) h'^2 (1 + a^2 h'^2)^s e_k(x). \end{aligned}$$

Hence equation (3) holds if and only if

$$-\sum_{k>0} \left[k^2 (1 + a^2 k^2)^s \frac{d\omega_k}{dt} + \frac{1}{2\pi} \sum_{\substack{h+h'=k, \\ h>0}} \omega_h \omega_{h'} (h \cdot h'^\perp) h'^2 (1 + a^2 h'^2)^s \right] e_k(x) = 0,$$

meaning that (3) can be written as an infinite dimensional ODE as follows:

$$\begin{aligned} 2\pi k^2 (1 + a^2 k^2)^s \frac{d\omega_k}{dt} &= - \sum_{\substack{h+h'=k, \\ h>0}} (h \cdot h'^\perp) h'^2 (1 + a^2 h'^2)^s \omega_h \omega_{h'} \\ &= \frac{1}{2} \sum_{\substack{h+h'=k, \\ h>0}} (h \cdot h'^\perp) [h^2 - h'^2] (1 + a^2 h'^2)^s \omega_h \omega_{h'}, \quad \forall k > 0. \end{aligned} \quad (4)$$

From $h'^\perp \cdot h = -h' \cdot h^\perp$ we obtain the following form of the averaged-Euler equations:

$$\frac{d\omega_k}{dt} = B_k(\varphi), \quad \forall k > 0,$$

where the vector field B is defined by

$$B(\varphi) = \sum_k B_k(\varphi) e_k, \quad (5)$$

with

$$B_k(\varphi) = \frac{1}{2\pi} \sum_{h>0} \left[\frac{1}{k^2} (h^\perp \cdot k) (h \cdot k) - \frac{1}{2} (h^\perp \cdot k) \right] \frac{(1 + a^2 (k-h)^2)^s}{(1 + a^2 k^2)^s} \omega_h \omega_{k-h}. \quad (6)$$

3. GAUSSIAN INVARIANT MEASURES FOR THE AVERAGED-EULER EQUATIONS

The purpose of this section is to construct an infinite dimensional Wiener measure μ_γ defined on some suitable $H^{p,s}$ space which is invariant for the averaged-Euler equation. We study the $L^p_{\mu_\gamma}$ regularity of the averaged-Euler vector field B , which is μ_γ -divergence free. This latter property will enable us to construct a flow associated with B . We proceed as in [2] where the Euler equation is considered.

Consider the probability measures on \mathbb{C} defined for $\gamma \in \mathbb{R}^+$ by

$$d\mu_{\gamma,k}(z) = \frac{\gamma k^4 (1 + a^2 k^2)^{2s}}{2\pi} \exp \left\{ -\frac{1}{2} \gamma k^4 (1 + a^2 k^2)^{2s} |z|^2 \right\} dx dy \quad (7)$$

where $z = x + iy$. Then

$$d\mu_\gamma(\varphi) = \prod_{k>0} d\mu_{\gamma,k}(\omega_k) \quad (8)$$

is a measure with support in $H^{1-\alpha,s}$ for any $\alpha > -\frac{s}{s+1}$; indeed,

$$\begin{aligned} \int \|\varphi\|_{1-\alpha,s}^2 d\mu_\gamma(\varphi) &= \sum_{k>0} k^{2(1-\alpha)} (1 + a^2 k^2)^{s(1-\alpha)} \prod_{h>0} \int |\omega_k|^2 d\mu_{\gamma,h}(\omega_h) \\ &= \frac{2}{\gamma} \sum_{k>0} \frac{1}{k^{2(1+\alpha)} (1 + a^2 k^2)^{s(1+\alpha)}} < +\infty, \quad \forall \alpha > -\frac{s}{s+1}. \end{aligned}$$

Proposition 3.1. *$(H^{1-\alpha,s}, H^{2,s}, \mu_\gamma)$ is a complex abstract Wiener space with measurable norm $\|\cdot\|_{1-\alpha,s}$ for any $\alpha > -\frac{s}{s+1}$.*

Proof. Consider the operator $\Gamma : H^{2,s} \rightarrow H^{2,s}$ defined by

$$\Gamma e_k = \frac{1}{|k|^{1+\alpha} (1 + a^2 k^2)^{(1+\alpha)s/2}} e_k,$$

which is a Hilbert-Schmidt operator since

$$\|\Gamma\|_{H.S.}^2 = \sum_{k>0} \frac{1}{k^{2(1+\alpha)} (1 + a^2 k^2)^{(1+\alpha)s}} < +\infty, \quad \forall \alpha > -\frac{s}{s+1}.$$

Let now $u = \sum_k u_k e_k$ in $H^{2,s}$, then

$$\|\Gamma u\|_{2,s}^2 = \sum_k k^{2(1-\alpha)} (1 + a^2 k^2)^{s(1-\alpha)} |u_k|^2 = \|u\|_{1-\alpha,s}^2.$$

Because Γ is a Hilbert-Schmidt operator such that $\|\Gamma u\|_{2,s}^2 = \|u\|_{1-\alpha,s}^2$ we can say that $\|\cdot\|_{1-\alpha,s}$ is a measurable norm in the sense of Gross (that is for every $\varepsilon > 0$ there exists $P_0 \in \mathcal{F}$, where \mathcal{F} is the partially ordered set of finite dimensional orthogonal projection P of the space $H^{2,s}$, such that $\mu_\gamma\{\|Pu\|_{1-\alpha,s} > \varepsilon\} < \varepsilon, \forall P \perp P_0 \in \mathcal{F}$). On the other hand $H^{1-\alpha,s}$ is the closure of $H^{2,s}$ with respect to the norm $\|\cdot\|_{1-\alpha,s}$, that is $(H^{1-\alpha,s}, H^{2,s}, \mu_\gamma)$ is a complex abstract Wiener space. Indeed μ_γ is a Wiener measure on $H^{1-\alpha,s}$, namely

$$\int e^{i\gamma l(u)} d\mu_\gamma(u) = e^{-\frac{1}{2}\gamma \|l\|_{2,s}^2}, \quad \forall l \in (H^{1-\alpha,s})' \subset H^{2,s}.$$

□

In particular $\mathbb{E}_{\mu_\gamma}(u_k) = 0$, $\mathbb{E}_{\mu_\gamma}(u_k \bar{u}_{k'}) = \frac{2\delta_{k,k'}}{\gamma k^4 (1 + a^2 k^2)^{2s}}$ and $\mathbb{E}_{\mu_\gamma}(|u_k|^{2p}) = \frac{2^p p!}{\gamma^p k^{4p} (1 + a^2 k^2)^{2sp}}$ for $p \geq 1$. For further studies on abstract Wiener spaces and results similar to Proposition 3.1 see [12].

3.1. Regularity of the vector field. We are looking for solutions of

$$\frac{d\omega_k}{dt} = B_k(\varphi), \quad \forall k > 0$$

that belong to $H^{1-\alpha,s}$ for all $t > 0$. Let us prove that $B : H^{1-\alpha,s} \rightarrow H^{1-\alpha,s}$, where B is defined in (5). We can consider finite dimensional approximations of B , namely

$$B^n(\varphi) = \sum_{k \in \{\alpha_1, \dots, \alpha_{d(n)}\}} B_k^n(\varphi) e_k(x), \quad (9)$$

that are vector fields on \mathbb{C}^d , $d = d(n)$, and write

$$B_k^n(\varphi) = \sum_{h^2 \leq n} \alpha_{h,k} \omega_h \omega_{k-h} \quad (10)$$

where $\alpha_i \in \mathbb{Z}^2$ for $i \in \{1, \dots, d\}$ and

$$\alpha_{h,k} = \frac{1}{2\pi} \left[\frac{1}{k^2} (h^\perp \cdot k)(h \cdot k) - \frac{1}{2} (h^\perp \cdot k) \right] \frac{(1 + a^2(k-h)^2)^s}{(1 + a^2 k^2)^s}.$$

Proposition 3.2. *The vector field $B \in L_{\mu_\gamma}^p(H^{1-\alpha,s}; H^{1-\alpha,s})$ for all $\alpha > 2$ and $p \geq 1$.*

Proof. It is sufficient to prove that $B \in L_{\mu_\gamma}^{2p}(H^{1-\alpha,s}; H^{1-\alpha,s})$ for all $\alpha > 2$ with p odd.

$$\begin{aligned} \mathbb{E}_{\mu_\gamma} \|B(\varphi)\|_{1-\alpha,s}^{2p} &= \mathbb{E}_{\mu_\gamma} \left(\sum_k k^{2(1-\alpha)} (1 + a^2 k^2)^{(1-\alpha)s} |B_k(\varphi)|^2 \right)^p \\ &= \mathbb{E}_{\mu_\gamma} \sum_{k_1, \dots, k_p} \prod_{i=1}^p k_i^{2(1-\alpha)} (1 + a^2 k_i^2)^{(1-\alpha)s} |B_{k_i}(\varphi)|^2 \\ &\leq \sum_{k_1, \dots, k_p} \prod_{i=1}^p k_i^{2(1-\alpha)} (1 + a^2 k_i^2)^{(1-\alpha)s} (\mathbb{E}_{\mu_\gamma} |B_{k_i}(\varphi)|^{2p})^{1/p} \\ &= \left[\sum_k k^{2(1-\alpha)} (1 + a^2 k^2)^{(1-\alpha)s} (\mathbb{E}_{\mu_\gamma} |B_k(\varphi)|^{2p})^{1/p} \right]^p. \end{aligned}$$

From $B_k(\varphi) = \sum_h \alpha_{h,k} \omega_h \omega_{k-h}$ we get that

$$\begin{aligned} \mathbb{E}_{\mu_\gamma} |B_k(\varphi)|^{2p} &= \mathbb{E}_{\mu_\gamma} \left(\sum_{\substack{h_1, \dots, h_p \\ h'_1, \dots, h'_p}} \prod_{i=1}^p \alpha_{h_i, k} \alpha_{h'_i, k} \omega_{h_i} \omega_{k-h_i} \bar{\omega}_{h'_i} \bar{\omega}_{k-h'_i} \right) \\ &\leq \left[\sum_{h, h'} \alpha_{h,k} \alpha_{h',k} (\mathbb{E}_{\mu_\gamma} (\omega_h \omega_{k-h} \bar{\omega}_{h'} \bar{\omega}_{k-h'})^p)^{1/p} \right]^p. \end{aligned}$$

Observe that, if $h \neq h'$ or $h \neq k - h'$,

$$\mathbb{E}_{\mu_\gamma} (\omega_h \omega_{k-h} \bar{\omega}_{h'} \bar{\omega}_{k-h'})^p = \mathbb{E}_{\mu_\gamma} (\omega_h)^p \mathbb{E}_{\mu_\gamma} (\omega_{k-h} \bar{\omega}_{h'} \bar{\omega}_{k-h'})^p$$

and for p odd $\mathbb{E}_{\mu_\gamma}(\omega_h)^p = 0$. Hence we have

$$\begin{aligned}
& \sum_{h, h' \neq 0, k} \alpha_{h,k} \alpha_{h',k} (\mathbb{E}_{\mu_\gamma}(\omega_h \omega_{k-h} \bar{\omega}_{h'} \bar{\omega}_{k-h'})^p)^{1/p} \\
&= \sum_{h, h' \neq 0, k} (\delta_{h,k-h'} \alpha_{h,k} \alpha_{h',k} + \delta_{h,h'} \alpha_{h,k} \alpha_{h',k}) (\mathbb{E}_{\mu_\gamma}(|\omega_h|^{2p} |\omega_{k-h}|^{2p}))^{1/p} \\
&\leq 2 \sum_{h \neq 0, k} |\alpha_{h,k}|^2 (\mathbb{E}_{\mu_\gamma} |\omega_h|^{2p})^{1/p} (\mathbb{E}_{\mu_\gamma} |\omega_{k-h}|^{2p})^{1/p} \\
&= 2 \sum_{h \neq 0, k} |\alpha_{h,k}|^2 \frac{(2^p p!)^{1/p}}{\gamma h^4 (1 + a^2 h^2)^{2s}} \frac{(2^p p!)^{1/p}}{\gamma (k-h)^4 (1 + a^2 (k-h)^2)^{2s}} \\
&\leq c(p, \gamma) \frac{1}{(1 + a^2 k^2)^{2s}} \sum_{h \neq 0, k} \left[\frac{1}{4h^2 (k-h)^2 (1 + a^2 h^2)^{2s}} \right]
\end{aligned}$$

and thus

$$\begin{aligned}
& \sum_k k^{2(1-\alpha)} (1 + a^2 k^2)^{(1-\alpha)s} (\mathbb{E}_{\mu_\gamma} |B_k(\varphi)|^{2p})^{1/p} \\
&\leq c(p, \gamma) \sum_k k^{2(1-\alpha)} (1 + a^2 k^2)^{-s(1+\alpha)} \sum_{h \neq 0, k} \left[\frac{1}{4h^2 (k-h)^2 (1 + a^2 h^2)^{2s}} \right]
\end{aligned}$$

where the series converge for $\alpha > 2$. \square

Corollary 3.1. *The convergence $\lim_{n \rightarrow +\infty} B^n = B$ holds in $L^2_{\mu_\gamma}(H^{1-\alpha,s}; H^{1-\alpha,s})$.*

Proof. To show this statement observe that

$$\mathbb{E}_{\mu_\gamma}(\|B^n(\varphi) - B(\varphi)\|_{1-\alpha,s}^2) = \sum_{k > 0} k^{2(1-\alpha)} (1 + a^2 k^2)^{(1-\alpha)s} \mathbb{E}_{\mu_\gamma}(|B_k^n(\varphi) - B_k(\varphi)|^2) \leq \varepsilon$$

for $\alpha > 2$ and n sufficiently big. In fact $\mathbb{E}_{\mu_\gamma}(|B_k^n(\varphi) - B_k(\varphi)|^2)$ is infinitesimal for n sufficiently big, since $\lim_{n \rightarrow +\infty} B_k^n(\varphi) = B_k(\varphi)$ for a.e $\varphi \in H^{1-\alpha,s}$ and B_k^n is a Cauchy sequence in $L^2_{\mu_\gamma}(H^{1-\alpha,s}; \mathbb{C})$, that is for $0 < n < m$

$$\begin{aligned}
\mathbb{E}_{\mu_\gamma} |B_k^n(\varphi) - B_k^m(\varphi)|^2 &= \sum_{\substack{n \leq h^2 \leq m \\ n \leq h'^2 \leq m}} \alpha_{h,h'} \alpha_{h',k} \mathbb{E}_{\mu_\gamma}(\omega_h \omega_{k-h} \bar{\omega}_{h'} \bar{\omega}_{k-h'}) \\
&= \frac{4}{\gamma^2} \sum_{\substack{n \leq h^2 \leq m \\ n \leq h'^2 \leq m}} \alpha_{h,h'} \alpha_{h',k} \frac{(\delta_{h,h'} + \delta_{h,k-h'})}{h^4 (1 + a^2 h^2)^{2s} (k-h)^4 (1 + a^2 (k-h)^2)^{2s}} \\
&\leq \frac{8}{\gamma^2} \sum_{n \leq h^2 \leq m} \frac{|\alpha_{h,k}|^2}{h^4 (1 + a^2 h^2)^{2s} (k-h)^4 (1 + a^2 (k-h)^2)^{2s}} < +\infty
\end{aligned}$$

as we saw above. \square

We shall consider the gradient operator in the sense of Malliavin calculus (c.f. [14]), that is, for $\psi : H^{1-\alpha,s} \rightarrow X$ where X is a Banach space, $\nabla \psi(u)$ is defined, for $u \in H^{1-\alpha,s}$,

by

$$\nabla\psi(u)(v) = D_v\psi(u) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} [\psi(u + \varepsilon v) - \psi(u)], \quad v \in H^{2,s}$$

where the limit is taken μ_γ -a.e. in $H^{1-\alpha,s}$. The second derivative is defined by iteration of the first, that is $\nabla^2\psi(u)(v, w) = D_v D_w \psi(u)$ for $u \in H^{1-\alpha,s}$ and $v, w \in H^{2,s}$, etc. Observe that the successive gradients $\nabla^r \psi(u)$ belong to $H_{\otimes^r}^{2,s}$ for $r \geq 1$. On the symmetric tensorial product $H_{\otimes^r}^{2,s} = H^{2,s} \otimes \dots \otimes H^{2,s}$ (r times) we consider the Hilbert-Schmidt norm.

For example, we can check that

$$D_{e_j} B(\varphi) = \sum_{k>0} (\alpha_{j,k} + \alpha_{k-j,k}) \omega_{k-j} e_k \quad (11)$$

and

$$D_{e_i} D_{e_j} B(\varphi) = \sum_{\substack{k=i+j, \\ k>0}} (\alpha_{j,k} + \alpha_{i,k}) e_k. \quad (12)$$

Hence given $\{\hat{e}_k\}_{k \in \mathbb{Z}^2}$ an orthonormal basis of $H^{2,s}$, namely $\hat{e}_k = \frac{e_k}{k^2(1+a^2k^2)^s}$, the Hilbert-Schmidt norms of ∇B and $\nabla^2 B$ are respectively

$$\begin{aligned} \|\nabla B(\varphi)\|_{H.S.}^2 &= \sum_j \|\nabla B(\varphi)(\hat{e}_j)\|_{1-\alpha,s}^2 \\ &= \sum_{j,k} \frac{k^{2(1-\alpha)}(1+a^2k^2)^{s(1-\alpha)}}{j^4(1+a^2j^2)^{2s}} (\alpha_{j,k} + \alpha_{k-j,k})^2 |\omega_{k-j}|^2 \end{aligned} \quad (13)$$

and

$$\begin{aligned} \|\nabla^2 B(\varphi)\|_{H.S.}^2 &= \sum_{i,j} \|D_{\hat{e}_i} D_{\hat{e}_j} B(\varphi)\|_{1-\alpha,s}^2 \\ &= \sum_{i,j} \frac{(i+j)^{2(1-\alpha)}(1+a^2(i+j)^2)^{(1-\alpha)s}}{i^4 j^4 (1+a^2i^2)^{2s} (1+a^2j^2)^{2s}} (\alpha_{j,i+j} + \alpha_{i,i+j})^2. \end{aligned} \quad (14)$$

Proposition 3.3. *For all $\alpha > 2$ and $p \geq 1$, $\nabla B \in L_{\mu_\gamma}^p(H^{1-\alpha,s}; H.S.(H^{2,s}, H^{1-\alpha,s}))$ and $\nabla^2 B \in L_{\mu_\gamma}^p(H^{1-\alpha,s}; H.S.(H^{2,s} \otimes H^{2,s}, H^{1-\alpha,s}))$.*

Proof.

$$\begin{aligned}
\mathbb{E}_{\mu_\gamma} \|\nabla B(\varphi)\|_{H.S.}^{2p} &= \mathbb{E}_{\mu_\gamma} (\|\nabla B(\varphi)\|_{H.S.}^2)^p \\
&= \mathbb{E}_{\mu_\gamma} \left[\sum_{j,k} \frac{k^{2(1-\alpha)} (1 + a^2 k^2)^{s(1-\alpha)}}{j^4 (1 + a^2 j^2)^{2s}} (\alpha_{j,k} + \alpha_{k-j,k})^2 |\omega_{k-j}|^2 \right]^p \\
&= \mathbb{E}_{\mu_\gamma} \sum_{\substack{k_1, \dots, k_p \\ j_1, \dots, j_p}} \prod_{i=1}^p \frac{k_i^{2(1-\alpha)} (1 + a^2 k_i^2)^{s(1-\alpha)}}{j_i^4 (1 + a^2 j_i^2)^{2s}} (\alpha_{j_i, k_i} + \alpha_{k_i - j_i, k_i})^2 |\omega_{k_i - j_i}|^2 \\
&\leq \sum_{\substack{k_1, \dots, k_p \\ j_1, \dots, j_p}} \prod_{i=1}^p \frac{k_i^{2(1-\alpha)} (1 + a^2 k_i^2)^{s(1-\alpha)}}{j_i^4 (1 + a^2 j_i^2)^{2s}} (\alpha_{j_i, k_i} + \alpha_{k_i - j_i, k_i})^2 (\mathbb{E}_{\mu_\gamma} |\omega_{k_i - j_i}|^{2p})^{1/p} \\
&= \left[\sum_{j,k} \frac{k^{2(1-\alpha)} (1 + a^2 k^2)^{s(1-\alpha)}}{j^4 (1 + a^2 j^2)^{2s}} (\alpha_{j,k} + \alpha_{k-j,k})^2 \frac{(2^p p!)^{1/p}}{\gamma (k-j)^4 (1 + a^2 (k-j)^2)^{2s}} \right]^p \\
&= \frac{(2^p p!)}{\gamma^p} C^p < +\infty
\end{aligned}$$

As we have shown in the proof of Proposition 3.2, the series above are convergent for every $\alpha > 2$ and $p \geq 1$. For the second derivative of B , the statement follows straightforward from the fact that (14) converges for every $\alpha > 2$ and $p \geq 1$. \square

3.2. The vector field is divergence free. Recall that on an abstract Wiener space (X, H, μ_γ) , the divergence of a vector field $\Psi : X \rightarrow G$, $\Psi \in L^2_{\mu_\gamma}(X; G)$, where G is a Hilbert space, is defined by

$$\int \delta_{\mu_\gamma} \Psi \cdot f d\mu_\gamma = \int (\Psi, \nabla f)_G d\mu_\gamma, \quad \forall f \in \mathcal{D} \quad (15)$$

where \mathcal{D} is the space of differentiable functions on X depending on a finite number of coordinates, that is $f(u) = f(u_{\alpha_1}, \dots, u_{\alpha_d})$ where $d = d(n)$ and $(\cdot, \cdot)_G$ is the inner product of G .

Following [2], where the Euler equation is treated, we show that the averaged-Euler vector field is μ_γ -divergence free.

Theorem 3.1. *For $\alpha > 2$ the vector field $B : H^{1-\alpha, s} \rightarrow H^{1-\alpha, s}$ defined above is divergence free with respect to the measure μ_γ , that is $\delta_{\mu_\gamma} B = 0$.*

Proof. Consider $d\mu_\gamma^n = \prod_{k \in \{\alpha_1, \dots, \alpha_d\}} d\mu_{\gamma, k}$ where $d = d(n)$ and denote by ρ_γ^n the density of this measure with respect to the Lebesgue measure. From the definition of divergence of a vector field and the fact that B^n converges to B in $L^2_{\mu_\gamma}(H^{1-\alpha, s}; H^{1-\alpha, s})$ when n goes to infinity, we have, for any $f \in \mathcal{D}$,

$$\begin{aligned}
\int \delta_{\mu_\gamma} B \cdot f d\mu_\gamma(\varphi) &= \int \langle B, \nabla f \rangle_{1-\alpha, s} d\mu_\gamma(\varphi) \\
&= \lim_n \int \sum_{k \geq 0} k^{2(1-\alpha)} (1 + a^2 k^2)^{s(1-\alpha)} B_k^n \overline{(\nabla f)_k} d\mu_\gamma^n(\varphi) = \lim_n \int \langle B^n \rho_\gamma^n, \nabla g \rangle_{\mathbb{C}^d} dz
\end{aligned}$$

where $g \in \mathcal{D}$ is defined by $g_k = k^{2(1-\alpha)}(1 + a^2 k^2)^{s(1-\alpha)} f_k$ for all $k \in \{\alpha_1, \dots, \alpha_{d(n)}\}$. Therefore, for all $g \in \mathcal{D}$, we have

$$\lim_n \int \operatorname{div}(B^n \rho_\gamma^n) g dz = \lim_n \int \left[\operatorname{div} B^n + \langle B^n, \frac{\nabla \rho_\gamma^n}{\rho_\gamma^n} \rangle_{\mathbb{C}^d} \right] g d\mu_\gamma^n(\varphi).$$

In particular,

$$\delta_{\mu_\gamma^n} B^n = \operatorname{div} B^n + \langle B^n, \frac{\nabla \rho_\gamma^n}{\rho_\gamma^n} \rangle_{\mathbb{C}^d} = 0,$$

since on one hand, from the definition of B^n (equations (9) and (10))

$$\operatorname{div} B^n(\varphi) = \sum_{k \in \{\alpha_1, \dots, \alpha_{d(n)}\}} D_{e_k} B_k^n(\varphi) = 0$$

and on the other hand,

$$\langle B^n, \frac{\nabla \rho_\gamma^n}{\rho_\gamma^n} \rangle_{\mathbb{C}^d} = -\gamma \langle B^n(\varphi), \varphi \rangle_{2,s} = 0$$

where the last equality holds by the conservation of the enstrophy. Therefore $\delta_{\mu_\gamma^n} B^n = 0$ for all $n \in \mathbb{N}$ and $\delta_{\mu_\gamma} B = 0$. \square

Using the fact that B belongs to $L^2_{\mu_\gamma}$ and has divergence zero (with respect to the measure μ_γ), it is possible to construct a flow associated to B for which μ_γ is an invariant measure.

Lemma 3.1. *There exists a unique solution of $\frac{dU^n(t, \varphi^n)}{dt} = B^n(U^n(t, \varphi^n))$, $U^n(0, \varphi^n) = \varphi^n(0)$ which is defined for all times.*

Proof. For each $k > 0$, $B_k^n(\varphi^n)$ is a finite sum of quadratic terms,

$$B_k^n(\varphi^n) = \sum_{h^2 \leq n} \alpha_{h,k} \varphi_h^n \varphi_{k-h}^n.$$

Then existence of a unique solution follows from classical results on ordinary differential equations, while the conservation of the energy ensure that the solution is globally defined in time. \square

Denote by $U^n(t, \varphi^n)$ the flow associated to B^n , that is $\varphi^n(0) \mapsto \varphi^n(t)$, and define the flow on $H^{1-\alpha,s}$ by

$$U^n(t, \varphi) = U^n(t, \varphi^n) + \Pi_n^\perp \varphi,$$

where $\Pi_n \varphi = \varphi^n$ stands for the orthogonal projection of u on the subspace spanned by $\{e_{\alpha_1}, \dots, e_{\alpha_{d(n)}}\}$, then we have

$$\frac{dU^n(t, \varphi)}{dt} = B^n(U^n(t, \varphi)), \quad U^n(0, \varphi) = \varphi(0),$$

in particular, $U^n(\cdot, \varphi) = \sum_k U_k^n(\cdot, \varphi) e_k$ where $U_k^n(\cdot, \varphi) \in C(\mathbb{R}; \mathbb{C})$ for all $k > 0$.

Theorem 3.2. *There exists a flow $U(t, \omega)$ defined on a probability space $(\Omega, \mathcal{F}, P_\gamma)$ with values in $H^{1-\alpha,s}$, $\alpha > 2$, $U(\cdot, \omega) \in C(\mathbb{R}; H^{1-\alpha,s})$, $\omega \in \Omega$ such that*

(i)

$$U_k(t, \omega) = U_k(0, \omega) + \int_0^t B_k(U(s, \omega)) ds, \quad P_\gamma - a.e. \omega, \quad \forall t \in \mathbb{R}, \quad (16)$$

(ii) and such that the measure μ_γ is invariant for the flow, in the sense that:

$$\int f(U(t, \omega)) dP_\gamma(\omega) = \int f d\mu_\gamma, \quad \forall t \in \mathbb{R}, \forall f \in \mathcal{D}. \quad (17)$$

Proof. The construction of such a flow can be found in [2] in the case of the two-dimensional Euler system. The same arguments apply in the case of the two-dimensional averaged-Euler equations. For $t \in \mathbb{R}^+$ consider $U_k^n(t, \varphi)$ as a stochastic process with law on $C(\mathbb{R}^+; \mathbb{C})$ defined by

$$\eta_k^n(\Gamma) = \mu_\gamma\{\varphi : U_k^n(\cdot, \varphi) \in \Gamma\}, \quad \Gamma \subset C(\mathbb{R}^+; \mathbb{C}).$$

Consider the sup-norm on $C(\mathbb{R}^+; \mathbb{C})$ and the weak topology on the space of measures over $C(\mathbb{R}^+; \mathbb{C})$. We have that:

1.

$$\eta_k^n(|y(0)| > R) \leq \frac{1}{R^2} \mathbb{E}_{\mu_\gamma}(|\omega_k|^2) = \frac{2}{\gamma R^2 k^4 (1 + a^2 k^2)^{2s}} \rightarrow 0 \quad \text{when } R \rightarrow +\infty$$

2. for all $\rho > 0$ and $T > 0$

$$\begin{aligned} \eta_k^n \left(\sup_{\substack{0 \leq t \leq t' \leq T \\ |t' - t| \leq \delta}} |y(t) - y(t')| > \rho \right) &\leq \frac{1}{\rho^2} \mathbb{E}_{\mu_\gamma} \left(\sup_{t, t'} |U_k^n(t, \varphi) - U_k^n(t', \varphi)|^2 \right) \\ &\leq \frac{\delta}{\rho^2} \mathbb{E}_{\mu_\gamma} \int_0^T |B_k^n(U^n(s, \varphi))|^2 ds \\ &\leq \frac{\delta T}{\rho^2} \mathbb{E}_{\mu_\gamma} |B_k^n|^2 \leq \frac{\delta T C}{\rho^2} \rightarrow 0 \quad \text{when } \delta \rightarrow 0, \end{aligned}$$

in the last inequalities we used respectively that $U^n(t, \varphi)$ is a flow for B^n and that B^n has null divergence with respect to μ_γ for all n . By 1. and 2. we are under the assumptions of Prohorov's criterium; then there exists a subsequence of η_k^n (again denoted by η_k^n) that converges weakly to η_k . Remark that we can choose an arbitrary subsequence since k belongs to \mathbb{Z}^2 that is countable. Hence, by Skorohod's theorem, there exists a probability space $(\Omega, \mathcal{F}, P_\gamma)$ and family of processes $U_k^n(t, \omega)$, $U_k(t, \omega)$, $\omega \in \Omega$, having laws respectively η_k^n and η_k on $C(\mathbb{R}^+; \mathbb{C})$. Furthermore, $U_k^n(\cdot, \omega) \rightarrow U_k(\cdot, \omega)$, P_γ -a.e. ω . Repeat for $t \in \mathbb{R}^+ \mapsto U_k^n(-t, \varphi)$ to get the negative values of t . We now prove (ii), take $f \in \mathcal{D}$,

$$\begin{aligned} \int f(U^n(t, \varphi)) d\mu_\gamma &= \int d\mu_\gamma^{n, \perp} \int f(U^n(t, \varphi^n)) d\mu_\gamma^n \\ &= \int d\mu_\gamma^{n, \perp} \int f d\mu_\gamma^n = \int f d\mu_\gamma, \quad \forall t > 0 \end{aligned}$$

where $d\mu_\gamma^n = \prod_{k \in \{\alpha_1, \dots, \alpha_{d(n)}\}} d\mu_{\gamma,k}$ and $d\mu_\gamma^{n,\perp} = \prod_{k \notin \{\alpha_1, \dots, \alpha_{d(n)}\}} d\mu_{\gamma,k}$. On the other hand denoting by η^n the law of $U^n(\cdot, \varphi)$, we also have

$$\begin{aligned} \int f d\mu_\gamma &= \int f(U^n(t, \varphi)) d\mu_\gamma = \int f(y(t)) d\eta^n \\ &= \int f(U^n(t, \omega)) dP_\gamma \rightarrow \int f(U(t, \omega)) dP_\gamma. \end{aligned}$$

Remark that $U(t, \omega)$ takes values in $H^{1-\alpha, s}$; in fact P_γ -a.e. $\omega \in \Omega$ we have

$$\int \|U(t, \omega)\|_{1-\alpha, s}^2 dP_\gamma = \int \|\varphi\|_{1-\alpha, s}^2 d\mu_\gamma < +\infty.$$

Finally we prove (i); we have

$$\begin{aligned} &\int \left| \int_0^t [B_k^n(U^n(s, \omega)) - B_k(U(s, \omega))] ds \right| dP_\gamma \\ &\leq \int \int_0^t |B_k^n(U^n(s, \omega)) - B_k(U^n(s, \omega))| ds dP_\gamma \\ &\quad + \int \int_0^t |B_k(U^n(s, \omega)) - B_k(U(s, \omega))| ds dP_\gamma. \end{aligned}$$

The first integral converges through zero by the identification in law of $U^n(t, \omega)$ and $U^n(t, \varphi)$, by the invariance of μ_γ under the flow $U^n(t, \varphi)$ and the fact that $B_k^n \rightarrow B_k$ in $L^2_{\mu_\gamma}$. The second integral converges towards zero by the dominated convergence theorem. Indeed $\{B_k(U^n(s, \omega))\}_{n \in \mathbb{N}^*}$ is uniformly integrable on $[0, t] \times \Omega$,

$$\int \int_0^t |B_k(U^n(s, \omega))| ds dP_\gamma = \int_0^t \int |B_k^n(\varphi)| d\mu_\gamma ds \leq \int_0^t \int |B_k(U(s, \omega))| dP_\gamma ds \leq Ct,$$

and $B_k(U_k^n(s, \omega)) \rightarrow B_k(U_k(s, \omega))$ P_γ -a.e. ω for all $s \in [0, t]$ when n goes to infinity. The latter statement follows from the fact that $U_k^n(\cdot, \omega) \rightarrow U_k(\cdot, \omega)$, P_γ -a.e. ω and that B_k^n converges uniformly to B_k (see Corollary 3.1) where

$$B_k(U^n(\cdot, \omega)) = \sum_{h^2 \leq n} \alpha_{h,k} U_h^n(\cdot, \omega) U_{k-h}^n(\cdot, \omega).$$

□

Remark 3.1. Uniqueness holds in the sense of Theorem 5.3 below, this implies a stronger result, namely

$$U(t, \varphi) = U(0, \varphi) + \int_0^t B(U(s, \varphi)) ds, \quad \mu_\gamma - a.e. \varphi, \quad \forall t \in \mathbb{R}.$$

See Remark 5.1 below.

Remark 3.2. At this point we could ask ourselves about the possibility of considering the (Gibbs) measure associated to the energy, formally

$$\nu_\gamma \simeq \frac{1}{Z} e^{-\frac{\gamma E}{2}} \times \text{“Lebesgue measure”},$$

where Z denotes a suitable normalizing constant, instead of the measure associated to the enstrophy μ_γ in (8). We can observe that the vector field B is not in L^2 with respect to ν_γ .

4. A SURFACE MEASURE

The energy of the averaged-Euler system belongs to the space $L^2_{\mu_\gamma}$. Therefore, as previously done in [8] for the Euler system (here a “renormalized” energy must be taken into account, because the energy is not square integrable with respect to the invariant measure), we consider the “surface” measure defined on the level sets of E , namely the conditional measure $\mu_\gamma(dx|E=r)$ for $r > 0$. We want to take advantage of the fact that the energy E is also a conserved quantity of the motion in order to construct a flow for the averaged-Euler vector field with values on the level sets of E .

Remark 4.1. It is not possible to construct a flow on the level sets of E using the invariant measure; in fact $\mu_\gamma\{\varphi|E(\varphi)=r\}=0$.

We consider suitable Sobolev spaces on $(H^{1-\alpha,s}, H^{2,s}, \mu_\gamma)$: the space W_1^p of the maps $f : H^{1-\alpha,s} \rightarrow \mathbb{R}$ that belong to $L^p_{\mu_\gamma}(H^{1-\alpha,s}; \mathbb{R})$ such that $\nabla f : H^{1-\alpha,s} \rightarrow H^{2,s}$, defined as $D_h f(x) = \langle \nabla f(x), h \rangle_{2,s}$ for all $h \in H^{2,s}$ satisfy $\nabla f \in L^p_{\mu_\gamma}(H^{1-\alpha,s}; H^{2,s})$. More generally the space W_r^p , for every integer $r > 1$, is the space of functions $f \in W_{r-1}^p$ such that $D_h f(x) \in W_{r-1}^p$ for all $h \in H^{2,s}$.

Proposition 4.1. *The energy E belongs to Sobolev spaces of all orders, that is $E \in W_\infty := \bigcap_{p,r} W_r^p$.*

Proof. First, we want to show that

$$(\mathbb{E}_{\mu_\gamma}|E(\varphi)|^{2m})^{1/m} < +\infty \quad \forall m = 2^{p-1} \text{ and } p \geq 2.$$

We have

$$(\mathbb{E}_{\mu_\gamma}|E(\varphi)|^{2m})^{1/m} \leq \sum_k \left[\mathbb{E}_{\mu_\gamma} (k^2(1+a^2k^2)^s |\omega_k|^2)^{2m} \right]^{1/m}$$

and

$$\mathbb{E}_{\mu_\gamma} (k^2(1+a^2k^2)^s |\omega_k|^2)^{2m} \leq c(p, \gamma) \frac{1}{k^{4m}(1+a^2k^2)^{2ms}};$$

thus

$$(\mathbb{E}_{\mu_\gamma}|E(\varphi)|^{2m})^{1/m} \leq c(p, \gamma) \sum_k \frac{1}{k^4(1+a^2k^2)^{2s}}.$$

Now consider the linear functional $\nabla E(\varphi) : H^{2,s} \rightarrow \mathbb{R}$,

$$\nabla E(\varphi)(e_k) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (E(\varphi + \varepsilon e_k) - E(\varphi)) = 2k^2(1+a^2k^2)^s |\omega_k|,$$

and take $\hat{e}_k = \frac{e_k}{k^2(1+a^2k^2)^s}$ for all $k > 0$, orthonormal basis of $H^{2,s}$; then $\nabla E(\varphi)(\hat{e}_k) = 2|\omega_k|$ and

$$(\mathbb{E}_{\mu_\gamma} \|\nabla E(\varphi)\|_{2,s}^{2m})^{1/m} \leq 4 \sum_k (\mathbb{E}_{\mu_\gamma} |\omega_k|^{2m})^{1/m} \leq c(p, \gamma) \sum_k \frac{1}{k^4(1+a^2k^2)^{2s}} < +\infty.$$

Finally observe that

$$(\mathbb{E}_{\mu_\gamma} \|\nabla^2 E(\varphi)\|_{H^{2,s} \otimes H^{2,s}}^{2m})^{1/m} = \sum_k \frac{1}{k^4(1+a^2k^2)^{2s}} < +\infty.$$

□

Next proposition is proved in [8], following [13], in the case of the Euler system.

Proposition 4.2. *E is of maximal rank, that is $\|\nabla E\|_{2,s}^{-1} \in W_\infty$.*

Proof. We want to show that $\mathbb{E}_{\mu_\gamma} \|\nabla E(\varphi)\|_{2,s}^{-2p} < +\infty$ for all p . By Chebycheff inequality, for all $t > 0$

$$-e^{-\frac{t}{\varepsilon}} \mathbb{E}_{\mu_\gamma} \left(e^{-t\|\nabla E(\varphi)\|_{2,s}^2} \right) \leq \mu_\gamma \{ \|\nabla E(\varphi)\|_{2,s}^2 \leq \varepsilon \} \leq e^{\frac{t}{\varepsilon}} \mathbb{E}_{\mu_\gamma} \left(e^{-t\|\nabla E(\varphi)\|_{2,s}^2} \right).$$

In particular

$$\mu_\gamma \{ \|\nabla E(\varphi)\|_{2,s}^2 \leq \varepsilon \} \geq -e^{-\frac{1}{\varepsilon}} \sum_{p \geq 0} \frac{(-1)^p}{p!} \mathbb{E}_{\mu_\gamma} (\|\nabla E(\varphi)\|_{2,s}^{-2p}),$$

meaning that $\mathbb{E}_{\mu_\gamma} \|\nabla E(\varphi)\|_{2,s}^{-2p}$ are finite for all p whenever $\mu_\gamma \{ \|\nabla E(\varphi)\|_{2,s}^2 \leq \varepsilon \}$ is finite. We have

$$\begin{aligned} \mu_\gamma \{ \|\nabla E(\varphi)\|_{2,s}^2 \leq \varepsilon \} &\leq e^{\frac{t}{\varepsilon}} \mathbb{E}_{\mu_\gamma} \left(e^{-t\|\nabla E(\varphi)\|_{2,s}^2} \right) \\ &= e^{\frac{t}{\varepsilon}} \prod_k \left(\frac{1}{1 + \frac{8t}{\gamma k^4(1+a^2k^2)^{2s}}} \right) \\ &\leq e^{\frac{t}{\varepsilon}} \prod_{\{k : \gamma k^4(1+a^2k^2)^{2s} < \frac{8}{t}\}} \left(\frac{1}{1+t^2} \right) \\ &\leq \inf_t e^{\frac{t}{\varepsilon}} \left(\frac{1}{1+t^2} \right)^c < +\infty \end{aligned}$$

where $c = \#\{k : \gamma k^4(1+a^2k^2)^{2s} < \frac{8}{t}\}$. □

For $g \in W_\infty$, we shall denote by $\rho(r) = \frac{d(E*\mu_\gamma)}{dr}$ and by $\rho_g(r) = \frac{d(E*g\mu_\gamma)}{dr}$ respectively the C^∞ densities of $d(E*\mu_\gamma)$ and $d(E*g\mu_\gamma)$ with respect to the Lebesgue measure, see [14, 1]. As proved in [1], Propositions 4.1 and 4.2 ensure the existence of a conditional measure of μ_γ knowing that $E = r$ for $r > 0$.

Theorem 4.1. *Let $r > 0$ be such that $\rho(r) > 0$; then there exists a Borel probability measure defined on $H^{1-\alpha,s}$, ν_γ^r , with support on $V_r = \{\varphi | E(\varphi) = r\}$ and such that*

$$\int g^*(\varphi) d\nu_\gamma^r = \frac{\rho_g(r)}{\rho(r)},$$

for any g^* redefinition of g .

Proof. See [1]. □

Remark 4.2. Recall that, given a measurable function Φ with values in \mathbb{R}^n , we call a (p, r) -redefinition of Φ a function Φ^* such that $\Phi = \Phi^*$ a.s. and Φ^* is (p, r) -continuous (that is, if $\forall \varepsilon > 0$ it is possible to find an open set O_ε such that $c_{p,r}(O_\varepsilon) < \varepsilon$ and the restriction of Φ^* to O_ε^c is continuous). The capacity of the open set O is given by $c_{p,r}(O) = \inf\{\|u\|_{W_{2r}^p}; u \geq 0, u(x) \geq 1, \mu - \text{a.e. on } O\}$; O is said to be slim if $c_{p,r}(O) = 0$, for all $p, r \in \mathbb{N}$.

For all $\Phi \in W_\infty$ there exists a redefinition Φ^* and a sequence of open sets $\{O_n\}_{n \in \mathbb{N}}$ associated to this redefinition such that: $\bigcap_n O_n$ is slim, Φ^* is continuous on $(\bigcap_n O_n)^c$ and Φ^* and $\nabla^r \Phi^*$ are continuous on O_n^c for all $n, r \in \mathbb{N}$.

The proof of Theorem 4.1 is essentially based on the following considerations. For a fixed $\Phi \in W_\infty$ of maximal rank and non-degenerate and for $g \in W_\infty$ we consider the map

$$\langle \delta \Phi, g \rangle : \xi \mapsto \langle \delta_\xi \Phi, g \rangle := \rho_g(\xi) / \rho(\xi);$$

this map belongs to $C^\infty(O; \mathbb{R})$ where $O = \{\xi \in \text{supp}(\Phi * \mu) \subset \mathbb{R}^n : \rho(\xi) > 0\}$. In particular the map

$$g \mapsto \langle \delta \Phi, g \rangle$$

is a continuous linear functional from W_∞ to the space of functions C^∞ on O . If $S(O)$ is the Schwartz space of O and W' the dual of W_∞ (W' was accurately defined by Watanabe, see [14]) we can consider the dual map

$$\delta_* \Phi : S(O) \rightarrow W'$$

that associates linear functionals on W_∞ to distributions over \mathbb{R}^n and such that

$$\langle \langle \delta_* \Phi, v \rangle, g \rangle = \langle v, \langle \delta \Phi, g \rangle \rangle \quad (18)$$

for every $v \in S(O)$ and $g \in W_\infty$. For further details see [14, 1].

We compute the second order moments of ν_γ^r . From [14] we know that $\rho_{\omega_k \bar{\omega}_{k'}} \in S(\mathbb{R})$ since $\omega_k \bar{\omega}_{k'} \in W_\infty$; then we have

$$\hat{\rho}_{\omega_k \bar{\omega}_{k'}}(r) = \int_{-\infty}^{+\infty} e^{ir\xi} \rho_{\omega_k \bar{\omega}_{k'}}(\xi) d\xi = \int_{H^{1-\alpha, s}} e^{irE(\varphi)} \omega_k \bar{\omega}_{k'} d\mu_\gamma.$$

If $k \neq k'$,

$$\begin{aligned} \hat{\rho}_{\omega_k \bar{\omega}_{k'}}(r) &= \prod_{j \neq k, k'} \int_{\mathbb{C}} e^{\frac{ir}{2} j^2 (1+a^2 j^2)^s |\omega_j|^2} d\mu_{\gamma, j} \int_{\mathbb{C}} e^{\frac{ir}{2} k^2 (1+a^2 k^2)^s |\omega_k|^2} \omega_k d\mu_{\gamma, k} \\ &\quad \int_{\mathbb{C}} e^{\frac{ir}{2} k'^2 (1+a^2 k'^2)^s |\omega_{k'}'|^2} \omega_{k'}' d\mu_{\gamma, k'} = 0, \end{aligned}$$

if $k = k'$,

$$\hat{\rho}_{\omega_k \bar{\omega}_{k'}}(r) = \prod_{j \neq k} \int_{\mathbb{C}} e^{\frac{ir}{2} j^2 (1+a^2 j^2)^s |\omega_j|^2} d\mu_{\gamma, j} \int_{\mathbb{C}} e^{\frac{ir}{2} k^2 (1+a^2 k^2)^s |\omega_k|^2} |\omega_k|^2 d\mu_{\gamma, k},$$

where

$$\begin{aligned}
& \int_{\mathbb{C}} e^{\frac{ir}{2}k^2(1+a^2k^2)^s|\omega_k|^2} |\omega_k|^2 d\mu_{\gamma,k} \\
&= \frac{\gamma k^4(1+a^2k^2)^{2s}}{2\pi} \int_{\mathbb{C}} e^{\frac{ir}{2}k^2(1+a^2k^2)^s|\omega_k|^2 - \frac{\gamma}{2}k^4(1+a^2k^2)^{2s}|\omega_k|^2} \omega_k \bar{\omega}_k dz \\
&= \frac{1}{\gamma k^4(1+a^2k^2)^{2s} - irk^2(1+a^2k^2)^s} \int_{\mathbb{C}} e^{\frac{ir}{2}k^2(1+a^2k^2)^s|\omega_k|^2} d\mu_{\gamma,k}
\end{aligned}$$

after complex by parts integration. Then

$$\hat{\rho}_{\omega_k \bar{\omega}_{k'}}(r) = \frac{1}{\gamma k^4(1+a^2k^2)^{2s} - irk^2(1+a^2k^2)^s} \int_{H^{1-\alpha,s}} e^{irE(\varphi)} d\mu_{\gamma}.$$

Hence

$$\begin{aligned}
\rho_{\omega_k \bar{\omega}_{k'}}(\xi) &= \frac{1}{k^2(1+a^2k^2)^s} \int_{-\infty}^{+\infty} e^{-ir\xi} \frac{1}{\gamma k^2(1+a^2k^2)^s - ir} \int_{H^{1-\alpha,s}} e^{irE(\varphi)} d\mu_{\gamma} dr \\
&= \frac{1}{k^2(1+a^2k^2)^s} \int_{-\infty}^{+\infty} e^{-ir\xi} \frac{\hat{\rho}(\xi)}{\gamma k^2(1+a^2k^2)^s - ir} dr
\end{aligned}$$

where

$$\begin{aligned}
\int_{-\infty}^{+\infty} e^{-ir\xi} \frac{\hat{\rho}(\xi)}{\gamma k^2(1+a^2k^2)^s - ir} dr &= \left(\hat{\rho}(\xi) \frac{1}{\gamma k^2(1+a^2k^2)^s - ir} \right)^{\vee} \\
&= \rho(\xi) * \left(2\pi e^{-\gamma k^2(1+a^2k^2)^s y} \right),
\end{aligned}$$

then

$$\rho_{\omega_k \bar{\omega}_{k'}}(\xi) = \frac{\pi}{k^2(1+a^2k^2)^s} \int_0^{+\infty} \rho(\xi + y) e^{-\gamma k^2(1+a^2k^2)^s y} dy.$$

We conclude that $\mathbb{E}_{\nu_{\gamma}^r}(\omega_k \bar{\omega}_{k'}) = \frac{\rho_{\omega_k \bar{\omega}_{k'}}(r)}{\rho(r)} = 0$ if $k \neq k'$ and $\mathbb{E}_{\nu_{\gamma}^r}(\omega_k \bar{\omega}_{k'}) = \frac{\pi}{k^2(1+a^2k^2)^s \rho(r)} \int_0^{+\infty} \rho(r+y) e^{-\gamma k^2(1+a^2k^2)^s y} dy$ if $k = k'$.

5. THE INVARIANT FLOW

5.1. Existence. Similar to [8], we show that the vector field B is divergence free with respect to the surface measure ν_{γ}^r . This will be fundamental for proving the existence of a flow on the level sets of the energy.

Theorem 5.1.

$$\int \langle B^n, \nabla f \rangle_{2,s}^* d\nu_{\gamma}^r = 0, \quad \forall f \in \mathcal{D}$$

for any $\langle B^n, \nabla f \rangle_{2,s}^*$ redefinition of $\langle B^n, \nabla f \rangle_{2,s}$.

Proof. Let $f \in \mathcal{D}$ and $v \in C_0^\infty(\mathbb{R})$ be arbitrary functions. We have,

$$\begin{aligned}
\int_{\mathbb{R}} v(r) \rho(r) \int_{V_r} \langle B^n, \nabla f \rangle_{2,s}^* d\nu_\gamma^r dr &= \int_{\mathbb{R}} v(r) d(E * \langle B^n, \nabla f \rangle_{2,s} \mu_\gamma) \\
&= \int_{H^{1-\alpha,s}} \langle v(E(\varphi)) B^n, \nabla f \rangle_{2,s} d\mu_\gamma \\
&= \int_{H^{1-\alpha,s}} \delta_{\mu_\gamma} (v(E(\varphi)) B^n) f d\mu_\gamma \\
&= \int_{H^{1-\alpha,s}} [v(E(\varphi)) \delta_{\mu_\gamma} B^n - v'(E(\varphi)) \langle B^n, \nabla E(\varphi) \rangle_{2,s}] f d\mu_\gamma \\
&= 0,
\end{aligned}$$

because, as we saw in Theorem 3.1, $\delta_{\mu_\gamma} B^n = 0$ and

$$\langle B^n, \nabla E(\varphi) \rangle_{2,s} = 2 \langle B(\varphi^n), \varphi^n \rangle_{1,s} = 0$$

since the energy is conserved. \square

In order to prove existence of a global averaged-Euler flow defined ν_γ^r almost everywhere and taking values on the level sets of the energy E , recall the finite dimensional result of Lemma 3.1 and that we denoted the flow associated to B^n on $H^{1-\alpha,s}$ by $U^n(t, \varphi) = U^n(t, \varphi^n) + \Pi_n^\perp \varphi$.

Theorem 5.2. *Let $\alpha > 2$. For all $r > 0$ such that $\rho(r) > 0$, there exists a flow $U'(\cdot, \omega)$ defined on a probability space $(\Omega, \mathcal{F}, P_\gamma^r)$ with values in V_r , $U'(\cdot, \omega) \in C(\mathbb{R}; V_r)$, $\omega \in \Omega$ such that:*

(i) *for any B^* redefinition of B ,*

$$U'_k(t, \omega) = U'_k(0, \omega) + \int_0^t B_k^*(U'(s, \omega)) ds, \quad P_\gamma^r - a.e. \omega, \quad \forall t \in \mathbb{R},$$

(ii) *ν_γ^r is invariant for the flow, in the sense that:*

$$\int f(U'(t, \omega)) dP_\gamma^r(\omega) = \int f(\varphi) d\nu_\gamma^r(\varphi), \quad \forall t \in \mathbb{R}, \quad \forall f \in \mathcal{D}.$$

Before the proof of the theorem we give a complementary Lemma.

Lemma 5.1. *The approximated averaged-Euler vector field B^n converges to B in $L_{\nu_\gamma^r}^2(H^{1-\alpha,s}; H^{1-\alpha,s})$ as $n \rightarrow \infty$.*

Proof. From the results on the regularity of B , Subsection 3.1, we get $B \in W_\infty(H^{1-\alpha,s})$ ($W_\infty(H^{1-\alpha,s})$ denotes the space of functions W_∞ with values in $H^{1-\alpha,s}$) and therefore

$$\int_{V_r} (\|B(\varphi)\|_{1-\alpha,s}^2)^* d\nu_\gamma^r < \infty.$$

Also from the results of Subsection 3.1 it follows that B^n is a Cauchy sequence in $W_r^p(H^{1-\alpha,s})$ for all r, p and by definition of ν_γ^r we have

$$\rho(r) \int_{V_r} (\|B^n - B\|_{1-\alpha,s}^2)^* d\nu_\gamma^r = \rho_{\|B^n - B\|_{1-\alpha,s}^2}^*(r).$$

Hence if we show that $\rho_{\|B^n - B\|_{1-\alpha,s}^2}(r)$ converges to zero as n tends to infinity, we get the lemma. From [1, 14] we know that $\rho_{\|B^n - B\|_{1-\alpha,s}^2}(r) \leq \| \|B^n - B\|_{1-\alpha,s}^2 \|_{W_r^p}$ while

$$\| \|B^n - B\|_{1-\alpha,s}^2 \|_{W_r^p} \leq C \|B^n - B\|_{W_r^{2p}(H^{1-\alpha,s})}^2.$$

In fact we have

$$|D_{\hat{e}_j} \|B^n - B\|_{1-\alpha,s}^2| \leq 2 \|D_{\hat{e}_j}(B^n - B)\|_{1-\alpha,s} \|B^n - B\|_{1-\alpha,s}$$

and thus

$$\|\nabla \|B^n - B\|_{1-\alpha,s}^2\|_{2,s} \leq 2 \|B^n - B\|_{1-\alpha,s} \|\nabla(B^n - B)\|_{H.S.(H^{2,s}; H^{1-\alpha,s})}.$$

A similar argument holds for the higher order derivatives. \square

In particular, from Lemma 5.1, there exists a constant C_2 such that

$$\sup_n \int_{V_r} (\|B^n(\varphi)\|_{1-\alpha,s}^2)^* d\nu_\gamma^r \leq C_2, \quad \forall \alpha > 2.$$

We finally prove Theorem 5.2

Proof of Theorem 5.2. Let $t \in \mathbb{R}^+$ and consider U_k^n as a stochastic process with laws on the space $C(\mathbb{R}^+; \mathbb{C})$ endowed with the sup-norm:

$$\eta_k^n(\Gamma) = \nu_\gamma^r(\{\varphi : U_k^n(\cdot, \varphi) \in \Gamma\}), \quad \Gamma \subset C(\mathbb{R}^+; \mathbb{C}).$$

We consider the weak topology on the space of measures on $C(\mathbb{R}^+; \mathbb{C})$. We have

1.

$$\eta_k^n(|y(0)| > R) \leq \frac{1}{R^2} \mathbb{E}_{\nu_\gamma^r} |\omega_k|^2 \leq \frac{C_3}{R^2} \rightarrow 0 \text{ when } R \rightarrow \infty$$

2. for all $L > 0$ and $T > 0$,

$$\begin{aligned} \eta_k^n \left(\sup_{\substack{0 \leq t \leq t' \leq T \\ t' - t > \delta}} |y(t') - y(t)| > L \right) &\leq \frac{1}{L^2} \mathbb{E}_{\nu_\gamma^r} \left(\sup_{t', t} |U_k^n(t', \varphi) - U_k^n(t, \varphi)|^2 \right) \\ &\leq \frac{\delta}{L^2} \mathbb{E}_{\nu_\gamma^r} \left(\int_0^T |B_k^n(U^n(s, \varphi))|^2 ds \right) \\ &\leq \frac{T\delta}{L^2} \mathbb{E}_{\nu_\gamma^r} |B_k^n|^2 \\ &\leq \frac{T\delta C_2}{L^2} \rightarrow 0 \text{ when } \delta \rightarrow 0, \end{aligned}$$

where in the last inequalities we used respectively that B^n is ν_γ^r -invariant for all n and that $\sup_n \mathbb{E}_{\nu_\gamma^r} \|B^n(\varphi)\|_{1-\alpha,s}^2 \leq C_2$. Hence, by Prohorov's criterium (actually a combined version of Prohorov's criterium and Ascoli-Arzelà theorem) we can state that there exists a subsequence of η_k^n (for simplicity it will also be denoted by η_k^n), that converges to some probability measure η_k . We denote by U_k the stochastic process with law η_k . By Skorohod's theorem, there exists a probability space $(\Omega, \mathcal{F}, P_\gamma^r)$ and a family of processes $U^n(t, \omega)$, $U'(t, \omega)$ with laws respectively η^n, η . Furthermore $U^n(\cdot, \omega) \rightarrow U'(\cdot, \omega)$ for a.e. $\omega \in \Omega$, that is, there exists $A \subset \Omega$ such that $P_\gamma^r(A^c) = 0$ and for all $\omega \in A$, $U^n(s, \omega) \rightarrow U'(s, \omega)$ in $H^{1-\alpha,s}$ for all $s \in [0, T]$. Repeating the construction for the

processes $t \in \mathbb{R}^+ \mapsto U_k^n(-t, \varphi)$ we obtain the negative values of t . We now prove (ii): for all f in \mathcal{D} , because $\delta_{\nu_\gamma^r} B^n = 0$ for all n (Theorem 5.1), we have,

$$\int f(U^n(t, \varphi)) d\nu_\gamma^r(\varphi) = \int f(\varphi) d\nu_\gamma^r(\varphi).$$

On the other hand, by definition of η^n and η ,

$$\begin{aligned} \int f(U^n(t, \varphi)) d\nu_\gamma^r(\varphi) &= \int f(y(t)) d\eta^n(y(t)) \\ &= \int f(U^n(t, \omega)) dP_\gamma^r(\omega) \\ &\rightarrow \int f(U'(t, \omega)) dP_\gamma^r(\omega) \text{ when } n \rightarrow \infty. \end{aligned}$$

To prove (i) it is enough to check that

$$\int \left| \int_0^t B_k^n(U^n(s, \omega)) - B_k^*(U'(s, \omega)) ds \right| dP_\gamma^r$$

converges to zero when n goes to infinity; we have

$$\begin{aligned} \int \left| \int_0^t B_k^n(U^n(s, \omega)) - B_k^*(U'(s, \omega)) ds \right| dP_\gamma^r &\leq \int_0^T \int |B_k^n(U^n(s, \omega)) - B_k^*(U^n(s, \omega))| dP_\gamma^r ds \\ &\quad + \int_0^T \int |B_k^*(U^n(s, \omega)) - B_k^*(U'(s, \omega))| dP_\gamma^r ds. \end{aligned}$$

The first integral converges to zero by the ν_γ^r -invariance of the flow and because B^n converges to B in $L_{\nu_\gamma^r}^2(H^{1-\alpha, s}; H^{1-\alpha, s})$ as we proved in Lemma 5.1. For the second integral consider $D' = [0, T] \times A^c$, clearly $\lambda \times P_\gamma^r(D') = 0$ where $\lambda(ds)$ denotes the Lebesgue measure on \mathbb{R} . Also we can find a subset $D \subset H^{1-\alpha, s}$ such that $\nu_\gamma^r(D) = 0$ and B^* restricted to D^c is continuous, then define

$$A_n = \{(s, \omega) \in [0, T] \times \Omega : U^n(s, \omega) \in D\}$$

and

$$A_\infty = \{(s, \omega) \in [0, T] \times \Omega : U'(s, \omega) \in D\}.$$

We have $\lambda \times P_\gamma^r(A_n) = 0$ for all n and $\lambda \times P_\gamma^r(A_\infty) = 0$. Set

$$\Delta = A_\infty \cup (\cup_n A_n) \cup D'.$$

Let $(s, \omega) \in \Delta^c$; in particular U^n and U' take values in D^c (in which B^* is continuous) and $U_k^n(s, \omega) \rightarrow U_k'(s, \omega)$ in \mathbb{C} . We have,

$$|B_k^*(U^n(s, \omega)) - B_k^*(U'(s, \omega))| \rightarrow 0$$

and since

$$\int_0^T \int |B_k^*(U^n(s, \omega)) - B_k^*(U'(s, \omega))| dP_\gamma^r ds$$

is uniformly bounded, by Egoroff's theorem the second integral also converges to zero. \square

5.2. Uniqueness. We are in the particular divergence free case of [4] in which also uniqueness of the flow is proved. All the regularity assumptions hold since $B \in L^2_{\nu_\gamma}$ and because of the following lemma.

Lemma 5.2. *For all $\alpha > 2$, $\nabla B \in L^2_{\nu_\gamma}(H^{1-\alpha,s}; H.S.(H^{2,s}; H^{1-\alpha,s}))$.*

Proof. We have

$$\begin{aligned} \mathbb{E}_{\nu_\gamma^r} \|\nabla B(\varphi)\|_{H.S.}^2 &= \sum_{j,k} \frac{k^{2(1-\alpha)}(1+a^2k^2)^{s(1-\alpha)}}{j^4(1+a^2j^2)^{2s}} (\alpha_{j,k} + \alpha_{k-j,k})^2 \mathbb{E}_{\nu_\gamma^r} |\omega_{k-j}|^2 \\ &= \sum_{j,k} \frac{k^{2(1-\alpha)}(1+a^2k^2)^{s(1-\alpha)}}{j^4(1+a^2j^2)^{2s}} (\alpha_{j,k} + \alpha_{k-j,k})^2 \frac{\pi \int_0^{+\infty} \frac{\rho(r+y)}{\rho(r)} e^{-\gamma(k-j)^2(1+a^2(k-j)^2)^s y} dy}{(k-j)^2(1+a^2(k-j)^2)^s}. \end{aligned}$$

Recall that for every fixed positive r , $\rho(r) = d(E * \mu_\gamma)/dr$, then we have

$$\int_0^{+\infty} \rho(r) dr = \int_{E^{-1}(0,+\infty)} d\mu_\gamma(\varphi),$$

while for all $y > 0$

$$\int_0^{+\infty} \rho(r+y) d(r+y) = \int_{E^{-1}(y,+\infty)} d\mu_\gamma(\varphi);$$

thus

$$\int_0^{+\infty} \rho(r+y) dr \leq \int_0^{+\infty} \rho(r) dr, \quad \forall y > 0.$$

Hence, using that $\sup_{y>0} \frac{\rho(r+y)}{\rho(r)} \leq 1$, we get

$$\begin{aligned} \mathbb{E}_{\nu_\gamma^r} \|\nabla B(\varphi)\|_{H.S.}^2 &\leq \sum_{j,k} \frac{k^{2(1-\alpha)}(1+a^2k^2)^{s(1-\alpha)}}{j^4(1+a^2j^2)^{2s}} (\alpha_{j,k} + \alpha_{k-j,k})^2 \frac{\pi \int_0^{+\infty} e^{-\gamma(k-j)^2(1+a^2(k-j)^2)^s y} dy}{(k-j)^2(1+a^2(k-j)^2)^s} \\ &\leq \sum_{j,k} \frac{k^{2(1-\alpha)}(1+a^2k^2)^{s(1-\alpha)}}{j^4(1+a^2j^2)^{2s}} (\alpha_{j,k} + \alpha_{k-j,k})^2 \frac{\pi}{\gamma(k-j)^4(1+a^2(k-j)^2)^{2s}} \\ &\leq \frac{\pi}{\gamma} C < +\infty. \end{aligned}$$

As we have shown in the proof of Proposition 3.3, the series above converges for every $\alpha > 2$. \square

Since each B -flow, that is a solution of $U(t, \omega) = \varphi + \int_0^t B^*(U(s, \omega)) ds$ for P_γ^r -a.e. ω , can be thought as a stochastic process with law over $C(\mathbb{R}; V_r)$, uniqueness of the flow follows from uniqueness of the corresponding law. The law η of a B -flow must be such that:

(i) η is concentrated on trajectories $y(t) \in C(\mathbb{R}; V^r)$ where

$$y(t) = y(0) + \int_0^t B^*(y(s)) ds \quad \forall t \in \mathbb{R};$$

(ii) $e_0 * \eta = \nu_\gamma^r$ where $e_0 : y(t) \mapsto y(0)$.

We reformulate the uniqueness result from [4] in our settings.

Theorem 5.3. *The flow $U(t, \omega)$ is unique in the sense that any other B -flow, $U'(t, \omega)$, is such that*

$$U(\cdot, \omega) = U'(\cdot, \omega), \quad P_\gamma^r - a.e. \omega \in \Omega.$$

Proof. Let $t \in \mathbb{R}^+$ and consider $U(t, \omega)$ and its law, η , on $C(\mathbb{R}^+; V_r)$. Denote by η_φ the measure η concentrated on the trajectories starting from $\varphi \in V_r$ at time zero, we want to show that the measures $\{\eta_\varphi\}_{\varphi \in V_r}$ are in fact Dirac masses. This reduces, as proved in [4], to show that for all fixed $\bar{t} \in \mathbb{Q} \cap \mathbb{R}^+$ the measures $e_{\bar{t}} * \eta_\varphi$ on V_r , where $e_t : y \mapsto y(t)$ is the evaluation map, are Dirac masses for ν_γ^r -a.e. $\varphi \in V_r$. Also from [4], we know that $\{e_{\bar{t}} * \eta_\varphi\}_{\varphi \in V_r}$ are Dirac masses if and only if for all disjoint Borelian $A_1, A_2 \subset V_r$

$$e_{\bar{t}} * \eta_\varphi(A_1) e_{\bar{t}} * \eta_\varphi(A_2) = 0, \quad \nu_\gamma^r - a.e. \varphi \in V_r.$$

For $\bar{t} = 0$ we have that $e_{\bar{t}} * \eta_\varphi = \delta_\varphi$, thus consider $\bar{t} > 0$. Suppose, by absurd, that there exists a Borel set $L \subset V_r$ with $\nu_\gamma^r(L) > 0$ and A_1, A_2 disjoint Borel sets in V_r such that $e_{\bar{t}} * \eta_\varphi(A_i) > 0$ for $i = 1, 2$ and $\varphi \in L$. Without loss of generality we can consider that $\beta(\varphi) := \frac{e_{\bar{t}} * \eta_\varphi(A_1)}{e_{\bar{t}} * \eta_\varphi(A_2)}$ is uniformly bounded in L . For $i = 1, 2$, let $\Omega_i \in C(\mathbb{R}^+; V_r)$ be the sets of trajectories, $y(t)$, which belong to A_i at time \bar{t} . Clearly $\Omega_1 \cap \Omega_2 = \emptyset$ and we can define positive finite measures η_i on $C(\mathbb{R}^+; V_r)$ by

$$\eta_1(\cdot) = \int_L \int \chi_{\Omega_1}(y(t)) d\eta_\varphi(y(t)) d\nu_\gamma^r(\varphi)$$

and

$$\eta_2(\cdot) = \int_L \int \beta(\varphi) \chi_{\Omega_2}(y(t)) d\eta_\varphi(y(t)) d\nu_\gamma^r(\varphi).$$

As proved in [4], both induce a weak solution, $e_t * \eta_i$, of the continuity equation

$$\frac{d(e_t * \eta_i)}{dt} + \operatorname{div}_{\nu_\gamma^r}(B e_t * \eta_i) = 0, \quad \text{in } \mathbb{R}^+ \times V_r.$$

Both solutions start from the same initial condition $e_{\bar{t}} * \eta_\varphi(A_1) \chi_L(\varphi)$, but, by definition of Ω_i , they are concentrated on two disjoint sets at time \bar{t} and thus uniqueness of solutions of the continuity equation is violated, contradicting [4]. To obtain the negative values of t , repeat the proof with $t \in \mathbb{R}^+ \mapsto U(-t, \omega)$.

Suppose now that σ , another measure on $C(\mathbb{R}; V_r)$, is the law of a process $U'(t, \omega)$ where $U'(t, \omega)$ is a B -flow. We just saw that the conditional measures σ_φ must be Dirac masses ν_γ^r -a.e. $\varphi \in V_r$. In this case $\frac{1}{2}\eta_\varphi + \frac{1}{2}\sigma_\varphi$ is also the law of a process that is a B -flow; indeed it is concentrated on trajectories $y(t) \in C(\mathbb{R}; V_r)$ where

$$y(t) = y(0) + \int_0^t B^*(y(s)) ds \quad \forall t \in \mathbb{R}$$

and $e_0 * (\frac{1}{2}\eta_\varphi + \frac{1}{2}\sigma_\varphi) = \nu_\gamma^r$. Therefore $\frac{1}{2}\eta_\varphi + \frac{1}{2}\sigma_\varphi$ must also be a Dirac mass ν_γ^r -a.e. $\varphi \in V_r$. Hence we have

$$\frac{1}{2}\eta_\varphi + \frac{1}{2}\sigma_\varphi = \frac{(\eta + \sigma)_\varphi}{2} \quad \nu_\gamma^r - a.e. \varphi \in V_r$$

and this is the case only if $\eta_\varphi = \sigma_\varphi$, ν_γ^r -a.e. $\varphi \in V_r$. We conclude that $U(\cdot, \omega)$ is the unique B -flow. \square

Remark 5.1. Since the law η_φ of the B -flow $U(t, \omega)$ is a Dirac mass on $C(\mathbb{R}; V^r)$, the flow is in fact deterministic; thus

$$U(t, \varphi) = \varphi + \int_0^t B^*(U(s, \varphi)) ds$$

holds ν_γ^r -a.e. $\varphi \in V^r$ and for all $t \in \mathbb{R}$. Moreover observe that uniqueness in the sense of Theorem 5.3 also holds for the flow constructed in Theorem 3.2, then

$$U(t, \varphi) = \varphi + \int_0^t B(U(s, \varphi)) ds$$

holds μ_γ^r -a.e. $\varphi \in H^{1-\alpha, s}$ and for all $t \in \mathbb{R}$.

5.3. Return to a neighborhood of its initial state. The Poincaré recurrence theorem holds in our particular case. This is used here to prove that the globally defined invariant flow returns infinitely many times in a neighborhood of the initial state. A similar result is proved in [9] for the one-dimensional Camassa-Holm equation and certain initial profiles for which the solutions exist globally. We recall the Poincaré recurrence theorem (c.f. [11]).

Theorem 5.4. *Let P be a probability measure defined on a set Ω . If $\{T_t\}_{t \geq 0}$ is a one-parameter family of measure preserving transformations and Ω_0 is a subset of Ω with $P(\Omega_0) > 0$, then for P -almost every $\omega \in \Omega_0$ there exist arbitrarily large t such that $T_t \omega \in \Omega_0$.*

Theorem 5.5. *Let $\alpha > 2$ and fix $\varphi_0 \in V_r \subset H^{1-\alpha, s}$. If $\varepsilon > 0$ is sufficiently small, then for ν_γ^r -a.e. $\varphi \in V_r \subset H^{1-\alpha, s}$ such that $\|\varphi - \varphi_0\|_{1-\alpha, s}^2 < \varepsilon$, there exists a sequence $\{t_n\} \uparrow \infty$ such that the corresponding invariant flow $U(t, \varphi)$ satisfies $\|U(t_n, \varphi) - \varphi_0\|_{1-\alpha, s}^2 < 2\varepsilon$.*

Proof. The statement follows by applying Poincaré recurrence theorem to the open set $B(\varphi_0, \varepsilon) = \{\varphi \in V_r \subset H^{1-\alpha, s} : \|\varphi - \varphi_0\|_{1-\alpha, s}^2 < \varepsilon\}$ with $\nu_\gamma^r\{B(\varphi_0, \varepsilon)\} > 0$. \square

Acknowledgements. The author wishes to thank Professor Ana Bela Cruzeiro for the introduction to the topic of invariant measures and for the great support during the preparation of this research. The author was funded by the LisMath fellowship PD/BD/52641/2014, FCT, Portugal.

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